Algorithms & Techniques for Dense Linear Algebra over Small Finite Fields

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ECrypt II PhD Summer School

Outline

- $\mathbb{F}_2$
  - Gray Codes
  - Multiplication
  - Elimination
- $\mathbb{F}_p$
- $\mathbb{F}_{2^e}$
  - Precomputation Tables
  - Karatsuba Multiplication
  - Performance
- $\mathbb{F}_p[x]$

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The M4RI Library

- available under the GPL Version 2 or later (GPLv2+)
- provides basic arithmetic (addition, equality testing, stacking, augmenting, sub-matrices, randomisation, etc.)
- asymptotically fast multiplication
- asymptotically fast elimination
- some multi-core support
- Linux, Mac OS X (x86 and PPC), OpenSolaris (Sun Studio Express) and Windows (Cygwin)

`http://m4ri.sagemath.org`

$\mathbb{F}_2$

- field with two elements.
- logical bitwise XOR is addition.
- logical bitwise AND is multiplication.
- 64 (128) basic operations in at most one CPU cycle
- ... arithmetic rather cheap

		$\oplus$	$\odot$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Memory access is the expensive operation, not arithmetic.

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Gray Codes

The Gray code [Gra53], named after Frank Gray and also known as reflected binary code, is a numbering system where two consecutive values differ in only one digit.

Gray Code Examples

0
1

0	**0**	$\Downarrow$
0	**1**	
1	1	
1	0	$\Uparrow$

0	**0**	**0**
0	**0**	**1**
0	**1**	**1**
0	**1**	**0**
1	1	0
1	1	1
1	0	1
1	0	0

0	**0**	**0**	**0**
0	**0**	**0**	**1**
0	**0**	**1**	**1**
0	**0**	**1**	**0**
0	**1**	**1**	**0**
0	**1**	**1**	**1**
0	**1**	**0**	**1**
0	**1**	**0**	**0**
1	1	0	0
1	1	0	1
1	1	1	1
1	1	1	0
1	0	1	0
1	0	1	1
1	0	0	1
1	0	0	0

Applications

Gray codes are used in various applications where all vectors over small finite fields need to be enumerated, such as:

- matrix multiplication;
- fast exhaustive search of Boolean polynomial systems;
- cube attacks on Grain-128.

Gray codes are a pretty basic part of the cryptographer's toolkit because they allow to reduce the cost of enumerating all vectors over $\mathbb{F}_2$ of length n from $n2^n - 1$ to $2^n - 1$.

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$\mathbb{F}_2$

Gray Codes

Multiplication

Elimination

$\mathbb{F}_p$

$\mathbb{F}_{2^e}$

Precomputation Tables

Karatsuba Multiplication

Performance

$\mathbb{F}_p[x]$

M4RM [ADKF70] I

Consider $C = A \cdot B$ (A is $m \times \ell$ and B is $\ell \times n$).

A can be divided into ℓ/k vertical "stripes"

$$A_0 \dots A_{(\ell-1)/k}$$

of k columns each. B can be divided into ℓ/k horizontal "stripes"

$$B_0 \dots B_{(\ell-1)/k}$$

of k rows each. We have:

$$C = A \cdot B = \sum_{0}^{(\ell-1)/k} A_i \cdot B_i.$$

M4RM [ADKF70] II

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_0 = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ 0 & 0 \\ \mathbf{1} & \mathbf{1} \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, B_0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_0 \cdot B_0 = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 0 & 1 & 1 & 0 \end{pmatrix}, A_1 \cdot B_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

M4RM: Algorithm $\mathcal{O}\left(n^3/\log n\right)$

```
begin
    C ⟵ create an m × n matrix with all entries 0;
    k ⟵ ⌊log n⌋;
    for 0 ≤ i < (ℓ/k) do
        // create table of 2^k − 1 linear combinations
        T ← MakeTable(B, i × k, 0, k);
        for 0 ≤ j < m do
            // read index for table T
            id ⟵ ReadBits(A, j, i × k, k);
            add row id from T to row j of C;
    return C;
```

Algorithm 1: M4RM

Strassen-Winograd [Str69] Multiplication

- fastest known pratical algorithm
- complexity: $\mathcal{O}\left(n^{\log_2 7}\right)$
- linear algebra constant: $\omega = \log_2 7$
- M4RM can be used as base case for small dimensions

$\rightarrow$ optimisation of this base case

Cache Friendly M4RM I

```
begin
    C ⟵ create an m × n matrix with all entries 0;
    for 0 ≤ i < (ℓ/k) do
        // this is cheap in terms of memory access
        T ← MakeTable(B, i × k, 0, k);
        for 0 ≤ j < m do
            // for each load of row j we take care of only k bits
            id ⟵ ReadBits(A, j, i × k, k);
            add row id from T to row j of C;
    return C;
```

Cache Friendly M4RM II

```
begin
    C ⟵ create an m × n matrix with all entries 0;
    for 0 ≤ start < m/b_s do
        for 0 ≤ i < (ℓ/k) do
            // we regenerate T for each block
            T ← MakeTable(B, i × k, 0, k);
            for 0 ≤ s < b_s do
                j ⟵ start × b_s + s;
                id ⟵ ReadBits(A, j, i × k, k);
                add row id from T to row j of C;
    return C;
```

$t > 1$ Gray Code Tables I

- actual arithmetic is quite cheap compared to memory reads and writes
- the cost of memory accesses greatly depends on where in memory data is located
- try to fill all of L1 with Gray code tables.
- Example: $k = 10$ and 1 Gray code table $\rightarrow$ 10 bits at a time. $k = 9$ and 2 Gray code tables, still the same memory for the tables but deal with 18 bits at once.
- The price is one extra row addition, which is cheap if the operands are all in cache.

$t > 1$ Gray Code Tables II

begin
    $C \longleftarrow$ create an $m \times n$ matrix with all entries 0;
    for $0 \leq i < (\ell/(2k))$ **do**
        $T_0 \leftarrow \textsc{MakeTable}(B, i \times 2k, 0, k)$;
        $T_1 \leftarrow \textsc{MakeTable}(B, i \times 2k + k, 0, k)$;
        for $0 \leq j < m$ **do**
            $id_0 \longleftarrow \textsc{ReadBits}(A, j, i \times 2k, k)$;
            $id_1 \longleftarrow \textsc{ReadBits}(A, j, i \times 2k + k, k)$;
            add row id_0 from T_0 and row id_1 from T_1 to row j of C;
    return C;

Performance: Multiplication

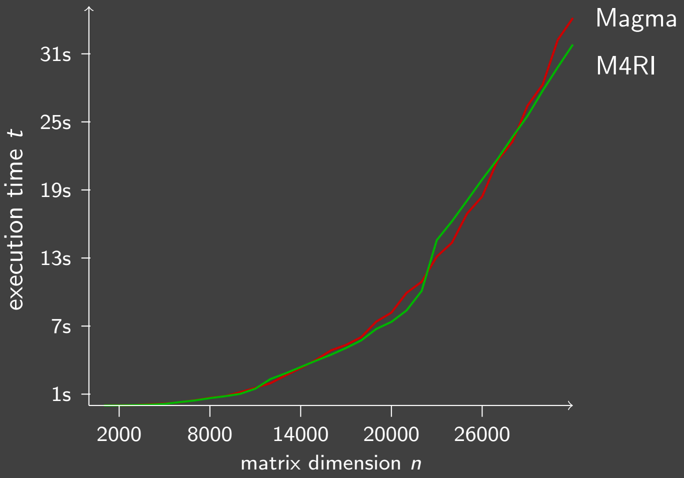

Figure: 2.66 Ghz Intel i7, 4GB RAM

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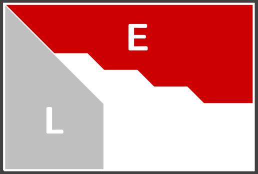
PLE Decomposition I



Definition (PLE)

Let A be a $m \times n$ matrix over a field K. A PLE decomposition of A is a triple of matrices P, L and E such that P is a $m \times m$ permutation matrix, L is a unit lower triangular matrix, and E is a $m \times n$ matrix in row-echelon form, and

$$A = PLE.$$

PLE decomposition can be in-place, that is L and E are stored in A and P is stored as an m-vector.

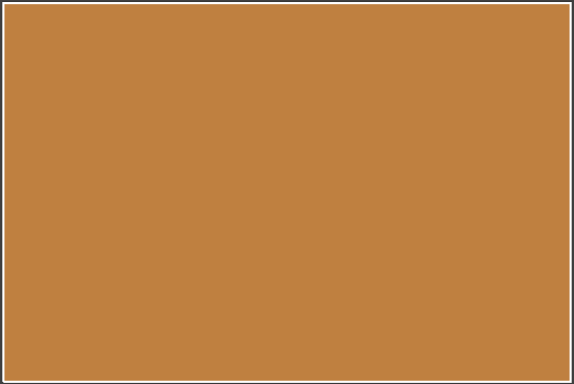
PLE Decomposition II

From the PLE decomposition we can

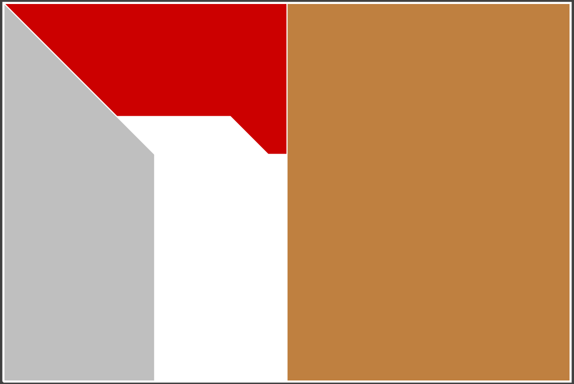
- read the rank r,
- read the row rank profile (pivots),
- compute the null space,
- solve $y = Ax$ for x and
- compute the (reduced) row echelon form.

C.-P. Jeannerod, C. Pernet, and A. Storjohann.
Rank-profile revealing Gaussian elimination and the CUP matrix decomposition.
arXiv:1112.5717, 35 pages, 2012.

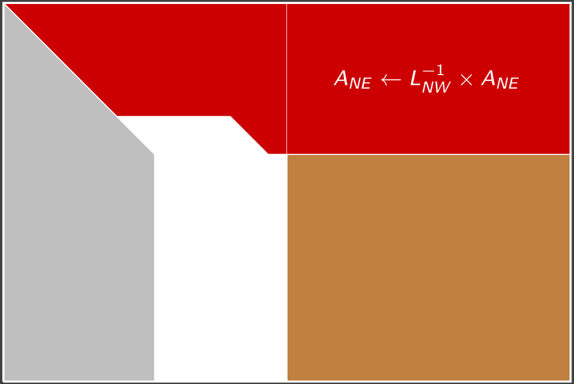
Block Recursive PLE Decomposition $\mathcal{O}(n^\omega)$ I



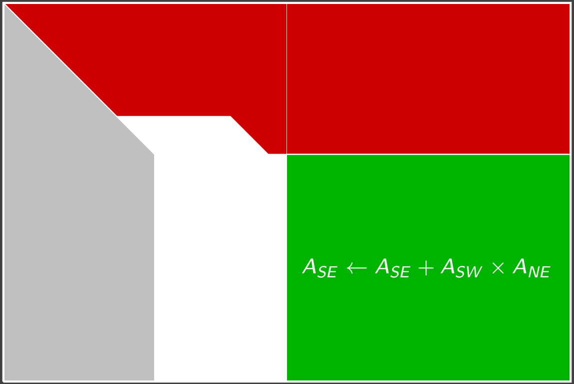
Block Recursive PLE Decomposition $\mathcal{O}(n^\omega)$ II



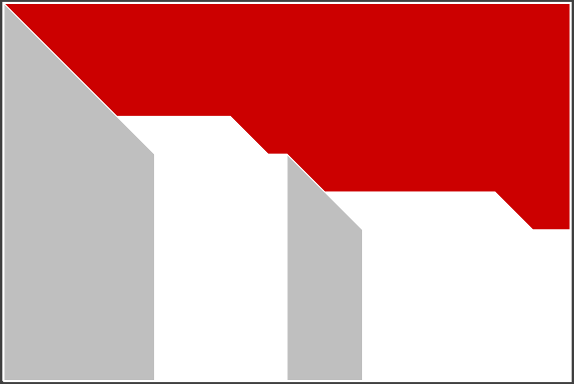
Block Recursive PLE Decomposition $\mathcal{O}(n^\omega)$ III



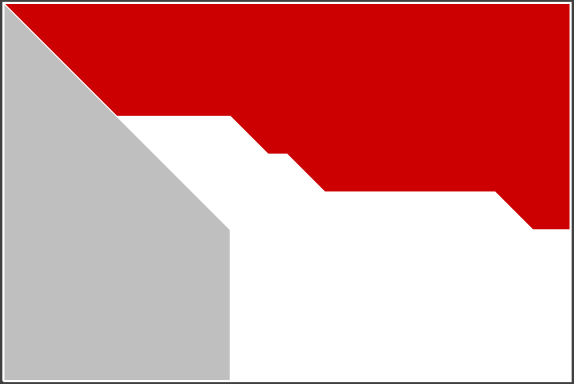
Block Recursive PLE Decomposition $\mathcal{O}(n^\omega)$ IV



Block Recursive PLE Decomposition $\mathcal{O}(n^\omega)$ V



Block Recursive PLE Decomposition $\mathcal{O}(n^\omega)$ VI



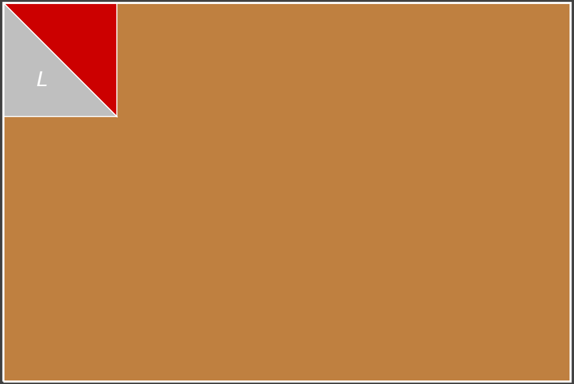
Block Iterative PLE Decomposition I

We need an efficient base case for PLE Decomposition

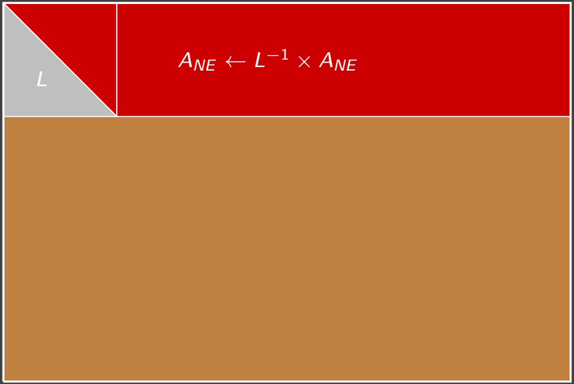
- block recursive PLE decomposition gives rise to a block iterative PLE decomposition
- choose blocks of size $k = \log n$ and use M4RM for the "update" multiplications
- this gives a complexity $\mathcal{O}\left(n^3/\log n\right)$

Block Iterative PLE Decomposition II

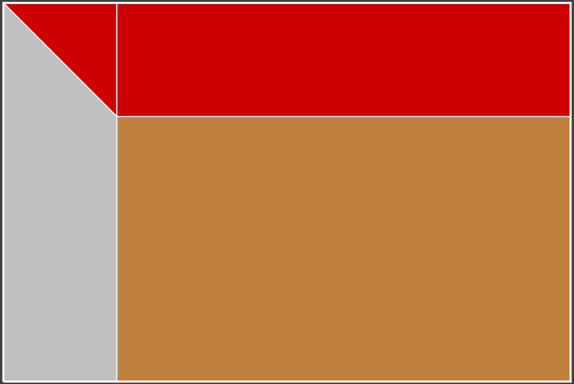
Block Iterative PLE Decomposition III



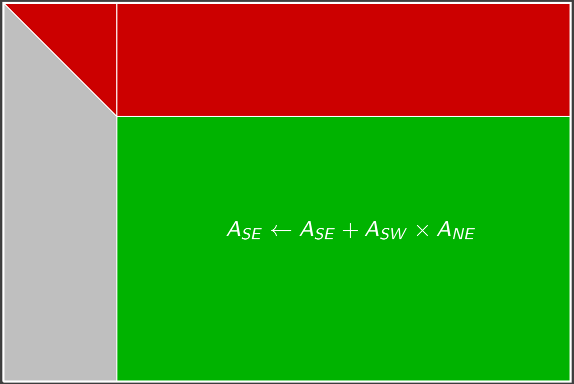
Block Iterative PLE Decomposition IV



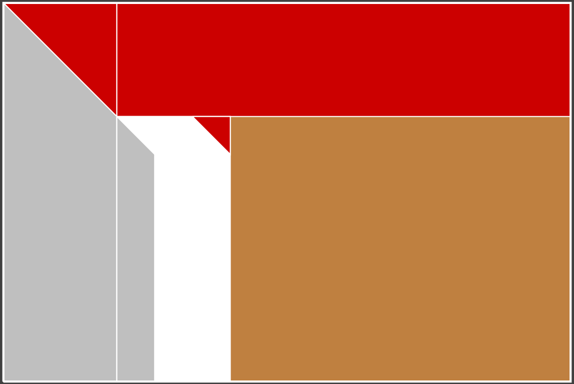
Block Iterative PLE Decomposition V



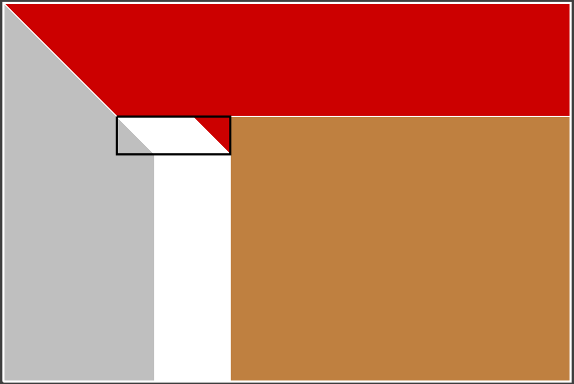
Block Iterative PLE Decomposition VI



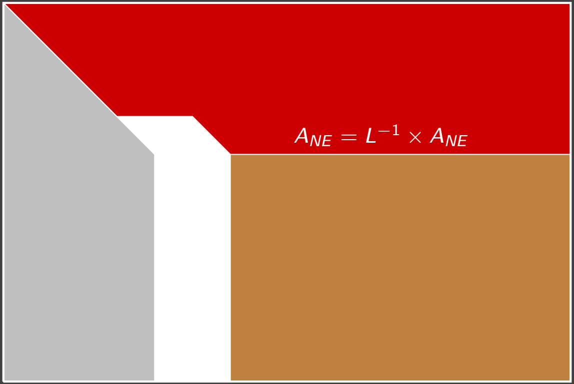
Block Iterative PLE Decomposition VII



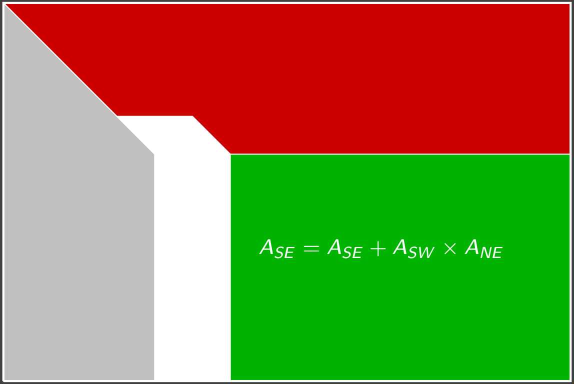
Block Iterative PLE Decomposition VIII



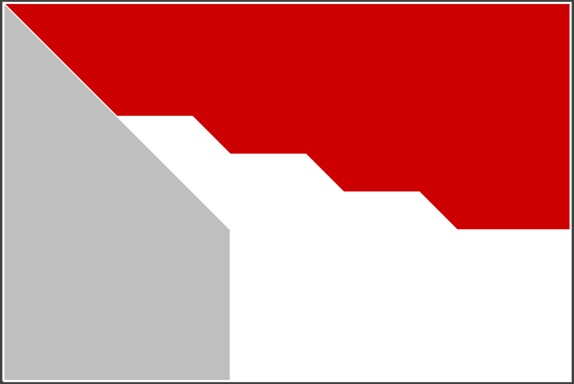
Block Iterative PLE Decomposition IX



Block Iterative PLE Decomposition X



Block Iterative PLE Decomposition XI



Performance: Reduced Row Echelon Form

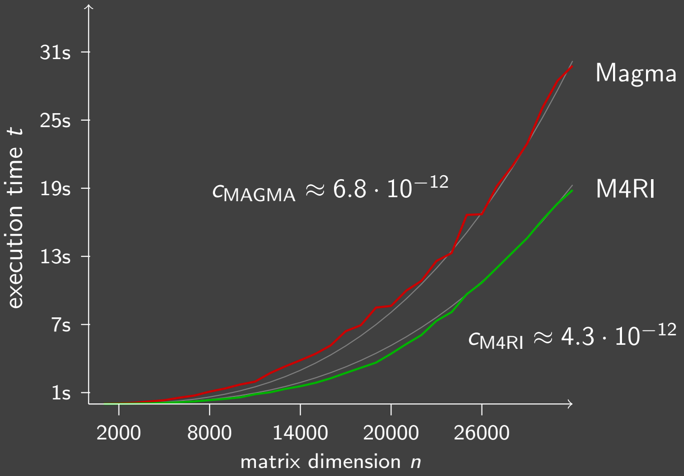


Figure: 2.66 Ghz Intel i7, 4GB RAM

Performance: Row Echelon Form

Using one core – on sage.math – we can compute the echelon form of a $500,000 \times 500,000$ dense random matrix over $\mathbb{F}_2$ in

$$9711 \text{ seconds } = 2.7 \text{ hours } (c \approx 10^{-12}).$$

Using four cores decomposition we can compute the echelon form of a random dense $500,000 \times 500,000$ matrix in

$$3806 \text{ seconds } = 1.05 \text{ hours.}$$

Caveat: Sensitivity to Sparsity

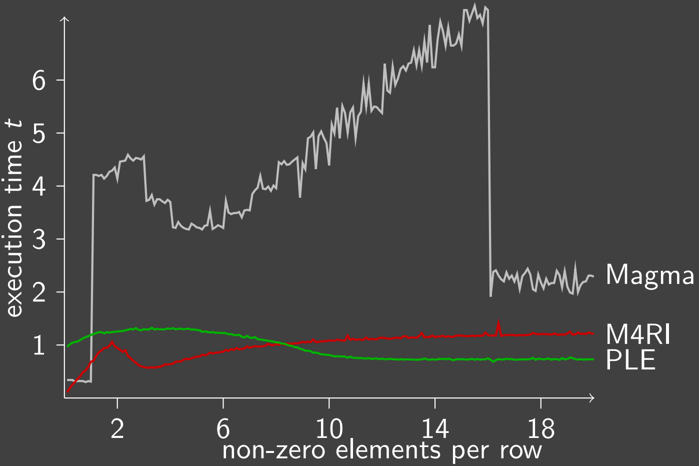
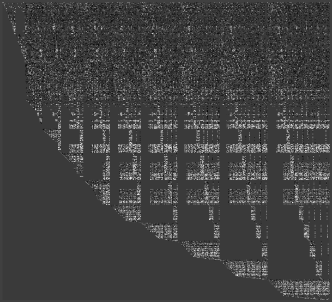


Figure: Gaussian elimination of $10,000 \times 10,000$ matrices on Intel 2.33GHz Xeon E5345 comparing Magma 2.17-12 and M4RI 20111004.

Caveat: Linear Algebra for Gröbner Basis



Problem	matrix dimensions	density	PLE	M4RI	GB
HFE 25 matrix 5 (5.1M)	12307 x 13508	0.07600	1.03	0.59	0.81
HFE 30 matrix 5 (16M)	19907 x 29323	0.06731	4.79	2.70	4.76
HFE 35 matrix 5 (37M)	29969 x 55800	0.05949	19.33	9.28	19.51
Mutant matrix (39M)	26075 x 26407	0.18497	5.71	3.98	2.10
random n=24, m=26 matrix 3 (30M)	37587 x 38483	0.03832	20.69	21.08	19.36
random n=24, m=26 matrix 4 (24M)	37576 x 32288	0.04073	18.65	28.44	17.05
SR(2,2,2,4) compressed, matrix 2 (328K)	5640 x 14297	0.00333	0.40	0.29	0.18
SR(2,2,2,4) compressed, matrix 4 (2.4M)	13665 x 17394	0.01376	2.18	3.04	2.04
SR(2,2,2,4) compressed, matrix 5 (2.8M)	11606 x 16282	0.03532	1.94	4.46	1.59
SR(2,2,2,4) matrix 6 (1.4M)	13067 x 17511	0.00892	1.90	2.09	1.38
SR(2,2,2,4) matrix 7 (1.7M)	12058 x 16662	0.01536	1.53	1.93	1.66
SR(2,2,2,4) matrix 9 (36M)	115834 x 118589	0.00376	528.21	578.54	522.98

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$p < 2^{23}$

- For medium sized primes your best bet is LinBox or more precisely FFLAS/FFPACK (C++ libraries).
- It reduces computations $\bmod p$ to computations with floating point numbers.
- On top of that it implements asymptotically fast techniques (Strassen, PLE, ...).

http://www.linalg.org/

p very small: Packing

- If p is small, you can pack several entries into one machine word. If there is enough zero padding these remain independent.
- There exists code to do this by the LinBox people but it's not in LinBox (yet).

p very small: Slicing

If $p \in (3, 5, 7)$ you can bit-slice your entries and implement the boolean circuit to perform arithmetic on machine words. If your prime has k-bits and you want to represent n elements, you'd represent your elements as k bitstrings of length n.

Example

Represent $\mathbb{F}_3$ as $0 : [0, 0], 1 : [1, 0], -1 : [1, 1]$. To add two elements $[x_0, x_1]$ and $[y_0, y_1]$ compute: $s \leftarrow x_0 \oplus y_1, t \leftarrow x_1 \oplus y_0$ and return $[s \wedge t, (s \oplus x_1) \vee (t \oplus y_1)]$.

Unfortunately, there is no ready-made library available yet which implements this (but there is some proof-of-concept code by Tom Boothby).

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$\mathbb{F}_p$

$\mathbb{F}_{2^e}$

- Precomputation Tables
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$\mathbb{F}_p[x]$

The M4RIE Library

- handles $\mathbb{F}_{2^e}$ for $2 \leq e \leq 10$; $e \leq 16$ planned.
- available under the GPL Version 2 or later (GPLv2+)
- provides basic arithmetic (addition, equality testing, stacking, augmenting, sub-matrices, randomisation, etc.)
- implements asymptotically fast multiplication
- implements asymptotically fast elimination
- Linux, Mac OS X (x86 and PPC), OpenSolaris, and Windows (Cygwin)

`http://m4ri.sagemath.org`

Representation of Elements I

Elements in $\mathbb{F}_{2^e} \cong \mathbb{F}_2[x]/f$ can be written as

$$a_0\alpha^0 + a_1\alpha^1 + \cdots + a_{e-1}\alpha^{e-1}.$$

We identify the bitstring $a_0, \ldots, a_{e-1}$ with

- the element $\sum_{i=0}^{e-1} a_i\alpha^i \in \mathbb{F}_{2^e}$ and
- the integer $\sum_{i=0}^{e-1} a_i 2^i$.

In the datatype `mzed_t` we pack several of those bitstrings into one machine word:

$$a_{0,0,0}, \ldots, a_{0,0,e-1},\ a_{0,1,0}, \ldots, a_{0,1,e-1},\ \ldots,\ a_{0,n-1,0}, \ldots, a_{0,n-1,e-1}.$$

Additions are cheap, scalar multiplications are expensive.

Representation of Elements II

- Instead of representing matrices over $\mathbb{F}_{2^e}$ as matrices over polynomials we may represent them as polynomials with matrix coefficients.
- For each degree we store matrices over $\mathbb{F}_2$ which hold the coefficients for this degree.
- The data type `mzd_slice_t` for matrices over $\mathbb{F}_{2^e}$ internally stores e-tuples of M4RI matrices, i.e., matrices over $\mathbb{F}_2$.

Additions are cheap, scalar multiplications are expensive.

Representation of Elements III

$$\begin{aligned} A &= \begin{pmatrix} \alpha^2+1 & \alpha \\ \alpha+1 & 1 \end{pmatrix} \\ &= \begin{bmatrix} \square 101 & \square 010 \\ \square 011 & \square 001 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right) \end{aligned}$$

Figure: 2×2 matrix over $\mathbb{F}_8$

Outline

The idea I

Input: A – $m \times n$ matrix
Input: B – $n \times k$ matrix

begin
  for $0 \leq i < m$ **do**
    for $0 \leq j < n$ **do**
      $C_j \longleftarrow C_j + A_{j,i} \times B_i$;
  return C;

The idea II

Input: A – $m \times n$ matrix
Input: B – $n \times k$ matrix

begin
  for $0 \leq i < m$ **do**
    for $0 \leq j < n$ **do**
      $C_j \longleftarrow C_j + A_{j,i} \times B_i$; // **cheap**
  return C;

The idea III

Input: A – $m \times n$ matrix
Input: B – $n \times k$ matrix

begin
  for $0 \leq i < m$ **do**
    for $0 \leq j < n$ **do**
      $C_j \longleftarrow C_j + A_{j,i} \times B_i$; // **expensive**
  return C;

The idea IV

Input: A – $m \times n$ matrix
Input: B – $n \times k$ matrix

begin
    for $0 \leq i < m$ **do**
        for $0 \leq j < n$ **do**
            $C_j \longleftarrow C_j + A_{j,i} \times B_i$; // **expensive**
    return C;

But there are only 2^e possible multiples of B_i.

The idea V

begin

    Input: A – $m \times n$ matrix

    Input: B – $n \times k$ matrix

    for $0 \leq i < m$ **do**

        for $0 \leq j < 2^e$ **do**

            $T_j \longleftarrow j \times B_i$;

        for $0 \leq j < n$ **do**

            $x \longleftarrow A_{j,i}$;

            $C_j \longleftarrow C_j + T_x$;

    return C;

$m \cdot n \cdot k$ additions, $m \cdot 2^e \cdot k$ multiplications.

Gaussian elimination & PLE decomposition

Input: A – $m \times n$ matrix

begin

    $r \longleftarrow 0$;
    for $0 \leq j < n$ **do**
        for $r \leq i < m$ **do**
            if $A_{i,j} = 0$ **then** **continue**;
            rescale row i of A such that $A_{i,j} = 1$;
            swap the rows i and r in A;
            $T \longleftarrow$ multiplication table for row r of A;
            for $r + 1 \leq k < m$ **do**
                $x \longleftarrow A_{k,j}$;
                $A_k \longleftarrow A_k + T_x$;
            $r \longleftarrow r + 1$;
        return r;

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The idea

- Consider $\mathbb{F}_{2^2}$ with the primitive polynomial $f = x^2 + x + 1$.
- We want to compute $C = A \cdot B$.
- Rewrite A as $A_0x + A_1$ and B as $B_0x + B_1$.
- The product is

$$C = A_0B_0x^2 + (A_0B_1 + A_1B_0)x + A_1B_1.$$

- Reduction modulo f gives

$$C = (A_0B_0 + A_0B_1 + A_1B_0)x + A_1B_1 + A_0B_0.$$

- This last expression can be rewritten as

$$C = ((A_0 + A_1)(B_0 + B_1) + A_1B_1)x + A_1B_1 + A_0B_0.$$

Thus this multiplication costs 3 multiplications and 4 adds over $\mathbb{F}_2$.

Outline

- $\mathbb{F}_2$
  - Gray Codes
  - Multiplication
  - Elimination
- $\mathbb{F}_p$
- $\mathbb{F}_{2^e}$
  - Precomputation Tables
  - Karatsuba Multiplication
  - Performance
- $\mathbb{F}_p[x]$

Performance: Multiplication

e	Magma 2.15-10	GAP 4.4.12	SW-NJ	SW-NJ/ M4RI	[Mon05]	Bitslice	Bitslice/ M4RI
1	0.100s	0.244s	–	1	1	0.071s	1.0
2	1.220s	12.501s	0.630s	8.8	3	0.224s	3.1
3	2.020s	35.986s	1.480s	20.8	6	0.448s	6.3
4	5.630s	39.330s	1.644s	23.1	9	0.693s	9.7
5	94.740s	86.517s	3.766s	53.0	13	1.005s	14.2
6	89.800s	85.525s	4.339s	61.1	17	1.336s	18.8
7	82.770s	83.597s	6.627s	93.3	22	1.639s	23.1
8	104.680s	83.802s	10.170s	143.2	27	2.140s	30.1

Table: Multiplication of $4,000 \times 4,000$ matrices over $\mathbb{F}_{2^e}$

Performance: Reduced Row Echelon Forms

e	Magma 2.15-10	GAP 4.4.12	LinBox (mod p) 1.1.6	M4RIE 6b24b839a46f
2	6.04s	162.65s	49.52s	3.31s
3	14.47s	442.52s	49.92s	5.33s
4	60.37s	502.67s	50.91s	6.33s
5	659.03s	N/A	51.20s	10.51s
6	685.46s	N/A	51.61s	13.08s
7	671.88s	N/A	53.94s	17.29s
8	840.22s	N/A	64.24s	20.25s
9	1630.38s	N/A	76.18s	260.77s
10	1631.35s	N/A	76.45s	291.30s

Table: Elimination of $10,000 \times 10,000$ matrices on 2.66Ghz i7

Outline

Prime-slicing

- The idea of bitsliced Karatsuba multiplication can be trivially extended to $\mathbb{F}_{p^e}$ and $\mathbb{F}_p[x]$ for $p > 2$.
- That is, we represent $(\mathbb{F}_p[x])^{m \times n}$ as $\mathbb{F}_p^{m \times n}[x]$ and
- use non-commutative Karatsuba-style formulas for multiplications in $\mathbb{F}_p[x]$.

Finding Formulas: Evaluation-Interpolation Schemes I

$f, g \in \mathbb{F}_{2^e}$, we

- consider them as polynomials $f(x), g(x)$ in $\mathbb{F}_2[x]$;
- evaluate those polynomials on sufficiently many points (possibly over some extension of $\mathbb{F}_2$),
- perform pointwise multiplication and
- interpolate $(f \cdot g)(x)$ from those points.

Finding Formulas: Evaluation-Interpolation Schemes II

Example: We multiply $f, g \in \mathbb{F}_{2^3}$, i.e., we are searching for

$$h(x) = f(x) \cdot g(x).$$

We compute $h(x)$ mod $p(x)$ where $\deg(p(x)) > \deg(h(x))$ such that $h(x) \bmod p(x) = h(x)$ and set

$$p(x) = (x + \infty) \cdot (x) \cdot (x + 1) \cdot (x^2 + x + 1).$$

That is, we compute modulo the factors of $p(x)$ and reconstruct the result using the Chinese remainder theorem. Multiplication modulo $(x + c)$ costs one in $\mathbb{F}_2$, modulo $x^2 + x + 1$ it costs 3 in $\mathbb{F}_2$. The total cost is 6 multiplications in $\mathbb{F}_2$.

Finding Formulas: Evaluation-Interpolation Schemes III

We can improve this strategy.

Example: We consider $f, g \in \mathbb{F}_{2^{11}}$. Instead of computing the solution modulo the product of **irreducible** polynomials

$$\begin{aligned} p(x) &= (x+\infty)\cdot(x)\cdot(x+1)\cdot(x^3+x+1)\cdot(x^3+x^2+1)\cdot \\ &\quad (x^4+x+1)\cdot(x^4+x^3+1)\cdot(x^4+x^3+x^2+x+1) \end{aligned}$$

with cost $3+2\cdot 6+3\cdot 9=42$, we compute modulo

$$\begin{aligned} p(x) &= (x+\infty)\cdot(\mathbf{x^2})\cdot(\mathbf{x+1})^2\cdot(x^2+x+1)\cdot(x^3+x+1)\cdot \\ &\quad (x^3+x^2+1)\cdot(x^4+x+1)\cdot(x^4+x^3+1). \end{aligned}$$

This only costs $1+3\cdot 3+2\cdot 6+2\cdot 9=40$ multiplications over $\mathbb{F}_2$.

Finding Formulas: Evaluation-Interpolation Schemes IV

How to find a good $p(x)$ for some degree e? $\Rightarrow$ We express this as a mixed integer linear program.

Let c be a table holding costs of polynomial multiplication, such that c_d is the cost of multiplying two polynomials modulo some polynomial of degree d: $c_0 = 0, c_1 = 1, c_2 = 3, \ldots$

Also, let

$$G_p(d) := \frac{1}{d} \sum_{d_i | d} \mu(d/d_i) p^{d_i}$$

be the function which returns the number of irreducible polynomials of degree d over $\mathbb{F}_p$.

Finding Formulas: Evaluation-Interpolation Schemes V

We want to minimize the function

$$1 + \sum_{d=1}^{\lceil \log_2(2e) \rceil} c_d n_d \qquad (1)$$

where n_d are number of degree d factors ($+1$ for $x + \infty$).

Our n_d must satisfy $\deg(p(x)) \geq 2e - 1$

$$\sum_{i=1}^{\lceil \log_2(2e) \rceil} n_d \cdot d \geq 2e - 2. \qquad (2)$$

Finding Formulas: Evaluation-Interpolation Schemes VI

We also have

$$0 \leq \sum_{i \in D(d)} n_i \leq \sum_{i \in D(d)} G_p(i) \tag{3}$$

for $1 \leq d \leq \lceil \log_2(2e) \rceil$ where $D(d)$ is defined as:

$$D(d) = \begin{cases} \{d\} & \text{if } d \text{ is odd} \\ \{d\} \cup D(d/2) & \text{else} \end{cases}$$

Minimizing (1) under the constraints (2) and (3), returns a $p(x)$ given by n_i.

This is a **very simple** mixed integer linear program and solving it for very large e is easy.

Finding Formulas: Evaluation-Interpolation Schemes VII

Adding a trick about field embeddings we get the follwing table.

e	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_{17}$	$\mathbb{F}_{39}$	$\mathbb{F}_{251}$
10	33	27	20	19	19
100	532	454	290	279	199
1000	6430	5455	3844	2997	2873
10000	71425	62845	43543	39217	29873
100000	755554	679861	474276	434007	355494

Table: Upper bounds on mul. in $\mathbb{F}_p$ for $f \cdot g \in \mathbb{F}_{p^e}$.

Note

There are sometimes better bounds known in the literature, the point here is that we can compute explicit formulas quickly.

Fin

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